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Schrödinger operators with potential $V(n) = n^{-\gamma}\cos(2\pi n^{\rho})$

Helge Krüger

ABSTRACT. Let H be the Schrödinger operator with potential $V(n) = n^{-\gamma}\cos(2\pi n^{\rho})$, where $\rho \in (1,2)$ and $\gamma \in (0, \frac{1}{2} - \frac{\rho-1}{2})$. I show that for almost every boundary condition H has pure-point spectrum.

1. Introduction

In this short note, I wish to show that for $1 < \rho < 2$ the sequence n^{ρ} (mod 1) generates sufficient randomness to obtain results for decaying potentials similar to the case of independent, identically distributed random variables as discussed by Simon in [**16**] and Kiselev, Last, and Simon in [**8**]:

Theorem 1.1 ([**8**]). *Let $\{\omega_n\}_{n=1}^{\infty}$ be independent, identically distributed, and non-constant random variables. Then for $\gamma \in (0, \frac{1}{2})$ the Schrödinger operator $\Delta + V$ on $\ell^2(\mathbb{Z}_+)$ with potential*

$$V(n) = \frac{\omega_n}{n^{\gamma}} \tag{1.1}$$

has pure point spectrum for almost every $\{\omega_n\}_{n=1}^{\infty}$.

Here $\Delta u(n) = u(n+1) + u(n-1)$ is the discrete Laplacian.

I will now introduce the operator studied in this paper and state the main result. Let ρ and γ satisfy

$$\rho \in (1,2), \quad \gamma \in \left(0, \frac{1}{2} - \frac{\rho - 1}{2}\right) \tag{1.2}$$

and introduce the potential

$$V_{\gamma,\rho}(n) = \frac{1}{n^{\gamma}}\cos(2\pi n^{\rho}). \tag{1.3}$$

For the boundary condition $\beta \in \mathbb{R}$, the half-line Schrödinger operator $H^{\beta}_{\gamma,\rho}$: $\ell^2(\mathbb{Z}_+) \to \ell^2(\mathbb{Z}_+)$ is given by

$$(H^{\beta}_{\gamma,\rho}u)(n) = \begin{cases} u(n+1) + u(n-1) + V_{\gamma,\rho}(n)u(n), & n \geq 2; \\ u(2) + (V_{\gamma,\rho}(1) + \beta)u(1), & n = 1. \end{cases} \tag{1.4}$$

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Since $V_{\gamma,\rho}(n) \to 0$ as $n \to \infty$, we have that $\sigma_{\mathrm{ess}}(H^{\beta}_{\gamma,\rho}) = [-2, 2]$.

Theorem 1.2. *Let ρ and γ obey (1.2). Then for almost every β, $H^{\beta}_{\gamma,\rho}$ has pure point spectrum in $[-2, 2]$.*

Arguments similar to the one used in this proof were first used by Kirsch, Molchanov, and Pastur [**6**], [**7**], then Stolz [**19**], and last Bourgain [**1**]. Bourgain's proof is closest to the one given in this paper.

This result is motivated by the work of Lukic [**13**], in which he showed that for $V(n) = \frac{1}{n^{\gamma}} \cos(2\pi\alpha n)$ with $\gamma > 0$ the operator H has absolutely continuous spectrum in $[-2, 2]$. Furthermore, he shows that the possible singular spectrum $[-2, 2]$ is contained in a finite set of points. That eigenvalues are possible is known since the work of Wigner and von Neumann on the continuous Schrödinger operator [**14**].

The condition on β in Theorem 1.2 is optimal. General results (see Section 12.4. in [**17**]) imply that for a dense G_{δ} set of β the operator $H^{\beta}_{\gamma,\rho}$ has singular continuous spectrum.

Remark 1.3. *It is possible to consider more general potentials of the form*

$$V(n) = \frac{1}{n^{\gamma}} \cos(2\pi n^{\rho}) + \frac{f(n)}{n^{\gamma+\varepsilon}}, \tag{1.5}$$

where γ and ρ obey (1.2), $f(n)$ is a bounded sequence, and $\varepsilon > 0$. The proof of the theorem is essentially the same.

It should also be possible to replace $x \mapsto \cos(2\pi x)$ with a sufficiently nice 1-periodic function. The main requirement is extending Theorem 3.1 on the spectrum of the Almost–Mathieu operator, whose proof is not the topic of this paper.

Remark 1.4. *The proof shows that for some $\varepsilon > 0$, we have for all generalized eigenfunctions ψ*

$$|\psi(n)| \le \exp(-|n|^{\varepsilon}) \tag{1.6}$$

for n large enough.

Let me now discuss, if the range of γ and ρ given in (1.2) is optimal. For simplicity, I restrict myself to $\gamma > 0$, when the potential is decaying

By the results of Christ and Kiselev [**2**], Deift and Killip [**4**], or Remling [**15**], we have that $H^{\beta}_{\gamma,\rho}$ has absolutely continuous spectrum for $\gamma > \frac{1}{2}$. Furthermore, the results of Stolz from [**18**] imply that $H^{\beta}_{\gamma,\rho}$ has purely absolutely continuous spectrum for $\rho \in (0, 1)$ and $\gamma > 0$. The already mentioned result by Lukic [**13**] imply absolutely continuous spectrum for $\rho = 1$ and $\gamma > 0$.

However, I would expect that in the complement of this range, that is $\rho > 1$ and $\gamma \in (0, \frac{1}{2})$ the operator $H^{\beta}_{\gamma,\rho}$ has pure point spectrum for almost every β. The main reason for this is that the Lyapunov exponent $L_{\lambda}(E)$ associated to the potential $V(n) = 2\lambda \cos(2\pi n^{\rho})$ is expected to behave like $L_{\lambda}(E) \ge \gamma\lambda^2$ for some $\gamma > 0$ as $\lambda \to 0$. See the work of Bourgain [**1**], and my own in [**10**] and [**11**] for some positive results in this direction.

If one compares Theorem 1.2 with the main result of [**16**], one notices that the result of this paper requires an almost sure choice of boundary condition, whereas [**16**] holds for almost every random parameter. It is an interesting question, if one could obtain a result similar to [16] in our context. In order to add a random parameter, one should modify the potential to

$$V_{\gamma,\rho,\vartheta}(n) = \frac{1}{n^\gamma}\cos(2\pi(n+\vartheta)^\rho). \tag{1.7}$$

Then the question is: Does Theorem 1.2 still hold with almost every β replaced by almost every ϑ? The reason for the choice of potential is that this implies some stability of the sets $I^\pm$ constructed in Proposition 3.2.

The rest of the paper splits into two sections. In the next one, I discuss some general properties of the resolvent equation. Then I use these in Section 3 to prove Theorem 1.2.

2. The resolvent equation

In this section, I will discuss a method to obtain bounds on the Green function. Let me begin by introducing the necessary notation. I will denote by $H = \Delta + V$ a Schrödinger operator either on $\ell^2(\mathbb{Z})$ or on $\ell^2(\mathbb{Z}_+)$. For $\Lambda \subseteq \mathbb{Z}$ denote by H^Λ the restriction of H to $\ell^2(\Lambda)$. I denote by $\{e_x\}_{x\in\mathbb{Z}}$ the standard basis of $\ell^2(\mathbb{Z})$, that is

$$e_x(n) = \begin{cases} 1, & x = n; \\ 0, & \text{otherwise.} \end{cases} \tag{2.1}$$

For $E \notin \sigma(H^\Lambda)$ and $x, y \in \Lambda$, the Green's function is defined by

$$G^\Lambda(E,x,y) = \langle e_x, (H^\Lambda - E)^{-1} e_y\rangle. \tag{2.2}$$

For either $x \notin \Lambda$ or $y \notin \Lambda$, we set $G^\Lambda(E,x,y) = 0$. For $\Xi \subseteq \Lambda$, I denote by χ_Ξ the restriction map $\ell^2(\Lambda) \to \ell^2(\Xi)$.

Let now $H = \Delta + V$ and $\widehat{H} = \Delta + \widehat{V}$ be Schrödinger operators and $I \subseteq \Lambda \cap \Xi$. A computation shows

$$(\widehat{H}^\Xi - E)^{-1}\chi_I - \chi_I(H^\Lambda - E)^{-1} = (\widehat{H}^\Xi - E)^{-1}(\chi_I H^\Lambda - \widehat{H}^\Xi\chi_I)(H^\Lambda - E)^{-1}. \tag{2.3}$$

If $\widehat{V}(n) = V(n)$ for $n \in I$, this formula becomes

$$(\widehat{H}^\Xi - E)^{-1}\chi_I - \chi_I(H^\Lambda - E)^{-1} = (\widehat{H}^\Xi - E)^{-1}[\chi_I, \Delta^\Xi](H^\Lambda - E)^{-1}, \tag{2.4}$$

where $[A,B] = AB - BA$ denotes the commutator. In the case of I an interval $[a,b]$ this commutator can be computed explicitly to be

$$[\chi_{[a,b]}, \Delta]u(n) = \begin{cases} -u(a), & n = a-1; \\ u(a-1), & n = a; \\ u(b+1), & n = b; \\ -u(b), & n = b+1; \\ 0, & \text{otherwise.} \end{cases} \tag{2.5}$$

Combining these considerations, we obtain

Lemma 2.1. *Let $x \in [a,b]$ and $y \in \Lambda \setminus [a,b]$. Assume $V(n) = \widehat{V}(n)$ for $n \in [a,b]$. Then*

$$|G^\Lambda(E,x,y)| \le \|(H^\Lambda - E)^{-1}\| \sum_{k\in\{a-1,a,b,b+1\}} |\widehat{G}^\Xi(E,k,x)|. \tag{2.6}$$

If $\widehat{H}$ is just defined on $\ell^2([a,b])$, then the equation becomes the more familiar

$$|G^\Lambda(E,x,y)| \le \|(H^\Lambda - E)^{-1}\| \sum_{k\in\{a,b\}} |\widehat{G}^{[a,b]}(E,k,x)|.$$

I should digress here a little bit, the computation done to prove Lemma 2.1 should be familiar from the context of continuum operators. In the discrete case, these are usually unnecessary, since everything is as smooth as one wants. However, in this work (2.6) is essential, since it allows us to avoid getting extra eigenvalues inside of gaps of operators, when restricting to some space interval. Using Lemma 2.1, we will show the following theorem.

Theorem 2.2. *Let H be a Schrödinger operator, $\varepsilon \in (0,1)$, $N \ge 1$ large enough, $\mathcal{E} \subseteq \mathbb{R}$, $c < \tilde{c} < \tilde{d} < d$ such that*

$$[c,d] = \Lambda_- \cup \Lambda_c \cup \Lambda_+$$

with $\Lambda_- = [c,\tilde{c}]$, $\Lambda_c = [\tilde{c}+1, \tilde{d}-1]$, and $\Lambda_+ = [\tilde{d}, d]$ disjoint intervals. Assume for $E \in \mathcal{E}$ that there exist $[a_\pm, b_\pm] \subseteq \Lambda_\pm$ and Schrödinger operators $H_\pm$ such that $b_\pm - a_\pm \ge N$,

$$\sigma(H_\pm) \cap [E - \frac{1}{N^{1-\varepsilon}}, E + \frac{1}{N^{1-\varepsilon}}] = \emptyset, \quad \|V - V_\pm\|_{\ell^\infty([a_\pm,b_\pm])} \le \frac{1}{2N^{1-\varepsilon}}. \tag{2.7}$$

Also assume

$$d - c \le \mathrm{e}^{N^{\frac{\varepsilon}{4}}}. \tag{2.8}$$

Then there exists $\mathcal{B} \subseteq \mathcal{E}$ such that

(i) $|\mathcal{B}| \le \mathrm{e}^{-N^{\frac{\varepsilon}{4}}}$.

(ii) *For $x \in \{c,d\}$, $y \in \Lambda_c$, and $E \in \mathcal{E} \setminus \mathcal{B}$, we have*

$$|G^{[c,d]}(E,x,y)| \le \mathrm{e}^{-N^{\frac{\varepsilon}{2}}}. \tag{2.9}$$

In words this theorem says *local gaps imply decay of the Green's function.* (2.7) makes precise what I mean by a *local gap*.

For the proof, we will need the Combes–Thomas estimate, [**3**]:

Lemma 2.3. *Let $\varepsilon > 0$. Then there exists $\kappa = \kappa(\varepsilon) > 0$, $\delta_0 = \delta_0(\varepsilon) > 0$ such that the following hold:*

For $E \in \mathbb{R}$, $\delta \in (0,\delta_0)$, $H : \ell^2(\Lambda) \to \ell^2(\Lambda)$ a Schrödinger operator with

$$\sigma(H) \cap [E-\delta, E+\delta] = \emptyset, \tag{2.10}$$

and $x, y \in \Lambda$ with $|x-y| \ge \frac{1}{\delta^{1+\varepsilon}}$, we have

$$|G(E,x,y)| \le \mathrm{e}^{-\kappa\delta|x-y|}. \tag{2.11}$$

Let $E \in \mathcal{E}$ and $a_\pm, b_\pm$ as in (2.7). Define

$$\widehat{H}_\pm = H_\pm + \chi_{[a_\pm,b_\pm]}(V - V_\pm). \tag{2.12}$$

Then by assumption $\|\widehat{H}_\pm - H_\pm\| \le \frac{1}{2N^{1-\varepsilon}}$ and thus

$$\sigma(\widehat{H}_\pm) \cap [E - \frac{1}{2N^{1-\varepsilon}}, E + \frac{1}{2N^{1-\varepsilon}}] = \emptyset. \tag{2.13}$$

By the Combes–Thomas estimate, we can conclude for $E \in I$,

$$y = \left\lfloor \frac{a_\pm + b_\pm}{2} \right\rfloor, \quad x \in \{a_\pm, b_\pm\}$$

that

$$|\widehat{G}_{\pm}^{[a_{\pm}, b_{\pm}]}(E, x, y)| \leq \mathrm{e}^{-\kappa N^{\varepsilon}} \tag{2.14}$$

as long as N is large enough. Here, $a_{\pm}, b_{\pm}$ depend on E, but κ does not.

Lemma 2.4. *Define $\mathcal{B}$ by*

$$\mathcal{B} = \bigcup_{a \in \Lambda_{-}, b \in \Lambda_{+}} \{E : \quad \operatorname{dist}(\sigma(H^{[a,b]}), E) \geq \mathrm{e}^{-N^{\frac{\varepsilon}{2}}}\} \tag{2.15}$$

Then $|\mathcal{B}| \leq \mathrm{e}^{-N^{\frac{\varepsilon}{4}}}$.

PROOF. The number of possible choices for a, b is bounded by $(d-c)^2$ and also $\#\sigma(H^{[a,b]}) \leq (d-c)$ for all possible choices of a, b. Hence

$$|\mathcal{B}| \leq 2(d-c)^3 \mathrm{e}^{-N^{\frac{\varepsilon}{2}}}.$$

The claim follows. □

PROOF OF THEOREM 2.2. Let $E \in I \setminus \mathcal{B}$. Then by applying (2.6) with $\Lambda = [c, d]$ and $[a, b] = [y_{-}, y_{+}]$, we can conclude that

$$|G^{[c,d]}(E, x, y)| \leq \mathrm{e}^{N^{\frac{\varepsilon}{4}}} \cdot (|G^{[c,d]}(E, x, y_{-})| + |G^{[c,d]}(E, x, y_{+})|).$$

Now using the result obtained by the Combes–Thomas estimate, and (2.6) once more, we obtain

$$|G^{[c,d]}(E, x, y)| \leq 8\mathrm{e}^{2N^{\frac{\varepsilon}{4}}} \cdot \mathrm{e}^{-\kappa N^{\varepsilon}}.$$

Choosing N large enough, the result follows. □

3. Proof of Theorem 1.2

Introduce for α irrational, $\omega \in [0, 1]$, and $\lambda > 0$ the Almost–Mathieu operator

$$\begin{aligned} &\widehat{H}_{\lambda,\alpha,\omega} : \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z}), \\ &\widehat{H}_{\lambda,\alpha,\omega} u(n) = u(n+1) + u(n-1) + 2\lambda \cos(2\pi(\omega + n\alpha)) u(n). \end{aligned} \tag{3.1}$$

We will need the following fact about the spectrum of this operator.

Theorem 3.1. *Let $\delta > 0$. There exists a constant $\kappa = \kappa(\delta) > 0$ and $\lambda_0 = \lambda_0(\delta) > 0$ such that for $\lambda \in (0, \lambda_0)$, $\omega \in [0, 1]$, and α satisfying*

$$2\cos(\pi\alpha) \in [-2 + \delta, -\delta] \cup [\delta, 2 - \delta], \tag{3.2}$$

we have

$$\operatorname{dist}(\pm 2\cos(\pi\alpha), \sigma(\widehat{H}_{\lambda,\alpha,\omega})) \geq \lambda\kappa. \tag{3.3}$$

PROOF. See **[1]**, **[5]**, **[10]**, **[12]**. □

For $k \geq 2$, we introduce the disjoint sets

$$\Lambda_k^c = [2^k, 2^{k+1}], \tag{3.4}$$

$$\Lambda_k^- = [2^{k-1}, 2^k - 1], \tag{3.5}$$

$$\Lambda_k^+ = [2^{k+1} + 1, 2^{k+1} + 2^{k-1}] \tag{3.6}$$

and $\Lambda_k = \Lambda_k^- \cup \Lambda_k^c \cup \Lambda_k^+$. We also define

$$\varepsilon = \frac{1}{6}(2 - \rho - 2\gamma). \tag{3.7}$$

If (1.2) holds, then $\varepsilon > 0$. Also note $\gamma + 3\varepsilon = 1 - \frac{\rho}{2} \in (0,1)$.

Proposition 3.2. *Let $\delta > 0$, $k \geq k_1(\delta)$ and $\alpha \in [0,1]$ with*

$$2\cos(\pi\alpha) \in [-2+\delta, -\delta] \cup [\delta, 2-\delta]. \tag{3.8}$$

Then there exist intervals $I^{\pm} \subseteq \Lambda_k^{\pm}$ satisfying

(i) $\#(I^{\pm}) \geq 2 \cdot 2^{(\gamma+2\varepsilon)k} + 1$.
(ii) $\|V_{\gamma,\rho} - V_{\lambda,\alpha,\omega}\|_{\ell^\infty(I_\pm)} = O(\frac{1}{2^{k(2-\rho)}})$ *for some ω and $\lambda \in [\frac{1}{2^{\gamma(k-1)}}, \frac{1}{2^{\gamma(k+2)}}]$.*

Proof. The arguments for I_- and I_+ are similar, so I restrict myself to I_-. Define $c = \lfloor \frac{2^{k-1}+2^k-1}{2} \rfloor$ and $\alpha_m = \rho m^{\rho-1}$. It is easy to check that

$$\alpha_{c+2^{\varepsilon k/4}} - \alpha_c \to \infty, \quad \frac{d}{dx}\rho x^{\rho-1} = O(\frac{1}{x^{2-\rho}}).$$

Using this, one concludes that for k large enough there exists $\hat{m}$ such that

$$|\hat{m} - c| \leq 2^{\varepsilon\frac{k}{2}}, \quad |\rho\hat{m}^{\rho-1} - \alpha| \leq \frac{4}{2^{(k-1)(2-\rho)}}.$$

Let $\ell = \lceil 2^{(\gamma+2\varepsilon)k} \rceil$ and define $I_- = [\hat{m} - \ell, \hat{m} + \ell]$. By some computations the claim follows. $\square$

By Theorem 2.2 with $N = 2^{(\gamma+2\varepsilon)k}$, we now obtain

Corollary 3.3. *Let $\delta > 0$ and $k \geq k_2(\delta)$. There exists a set $\mathcal{E}_{\delta,k}$ such that*

(i) $|\mathcal{E}_{\delta,k}| \leq \frac{1}{k^2}$.
(ii) *Let $y \in \Lambda_k^c$, $x \in \{2^{k-1}, 2^{k+1} + 2^{k-1}\}$, and*

$$E \in ([-2+\delta, -\delta] \cup [\delta, 2-\delta]) \setminus \mathcal{E}_{\delta,k}. \tag{3.9}$$

We have

$$|G_{\Lambda_k}(E, x, y)| \leq \exp\left(-2^{\frac{1}{5}\varepsilon k}\right). \tag{3.10}$$

We now proceed to derive Theorem 1.2. The strategy of proof is often called *spectral averaging*. See for example Section 12.3. in the book [**17**] by Simon for another implementation of this strategy. Fix some γ, ρ satisfying (1.2). For $\delta > 0$, introduce

$$\mathcal{E}_\delta = \bigcap_{\ell \geq k_2(\delta)} \left(\bigcup_{k \geq \ell} \mathcal{E}_{\delta,k} \right). \tag{3.11}$$

The Borel–Cantelli argument shows that $|\mathcal{E}_\delta| = 0$. In particular $\mathcal{E} = \bigcup_{j\geq 2} \mathcal{E}_{\frac{1}{j}}$ also has zero measure. It is well-known that there exists a unique probability measure μ^β that satisfies

$$\int \frac{1}{t-z} d\mu^\beta(t) = \langle e_1, (H^\beta_{\gamma,\rho} - z)^{-1} e_1 \rangle \tag{3.12}$$

for $\operatorname{Im}(z) > 0$. This measure is known a the *spectral measure*.

Lemma 3.4. *There exists a set $\mathcal{B}$ such that $|\mathbb{R} \setminus \mathcal{B}| = 0$ and for $\beta \in \mathcal{B}$, we have $\mu^\beta(\mathcal{E}) = 0$.*

PROOF. By Theorem 11.8. in [17], we have that $\int \mu^{\beta} d\beta$ is the Lebesgue measure. Thus

$$\int \mu^{\beta}(\mathcal{E}) d\beta = 0.$$

Since $\mu^{\beta}(\mathcal{E}) \geq 0$, the claim follows. □

PROOF OF THEOREM 1.2. Let $\beta \in \mathcal{B}$. For μ^{β} almost every $E \in (-2,2) \setminus \{0\}$ there exists a generalized eigenfunction (see Lemma 3.1. in [20]), that is a nonzero solution u of $H^{\beta}_{\gamma,\rho} u = Eu$ satisfying $|u(n)| \leq n$ for $n \geq 1$ and $u(0) = 0$. If we show that u is in $\ell^2(\mathbb{Z}_+)$, we obtain that μ^{β} almost every E is an eigenvalue, thus that μ^{β} is pure point.

By construction of $\mathcal{E}$, we can choose $\delta > 0$ and $\ell \geq 1$ such that

$$E \in [-2+\delta, -\delta] \cup [\delta, 2-\delta].$$

and for $k \geq \ell$

$$E \in \mathcal{E}_{\delta,k}.$$

For $x \in \Lambda_k^c = [2^k, 2^{k+1}]$, we have

$$u(x) = -G^{\Lambda_k}_{\gamma,\rho}(E, x, 2^k) u(2^{k-1} - 1) - G^{\Lambda_k}_{\gamma,\rho}(E, x, 2^{k+1} + 2^{k-1}) u(2^{k+1} + 2^{k-1} + 1).$$

Corollary 3.3, we obtain that $|u(x)| \leq \frac{1}{x^2}$ for k large enough. This implies that $u \in \ell^2(\mathbb{Z}_+)$ finishing the proof. □

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References

[1] J. Bourgain, *Positive Lyapounov exponents for most energies*, Geometric aspects of functional analysis, 37–66, Lecture Notes in Math. **1745**, Springer, Berlin, 2000.

[2] M. Christ, A. Kiselev, *Absolutely continuous spectrum for one-dimensional Schrödinger operators with slowly decaying potentials: some optimal results*. Journal of the AMS, **11** (1998), 771–797.

[3] J.M. Combes, L. Thomas, *Asymptotic behaviour of eigenfunctions for multiparticle Schrödinger operators*. Comm. Math. Phys. **34** (1973), 251–270.

[4] P. Deift, R. Killip, *On the absolutely continuous spectrum of one-dimensional Schrödinger operators with square summable potentials*. Comm. Math. Phys. 203, 341 347 (1999).

[5] B. Helffer, P. Kerdelhué, J. Sjöstrand, *Le papillon de Hofstadter revisité*. Mém. Soc. Math. France (N.S.), No. 43 (1990), 87 pp.

[6] W. Kirsch, S.A. Molchanov, L.A. Pastur, *The one-dimensional Schrdinger operator with unbounded potential: the pure point spectrum*. Funktsional. Anal. i Prilozhen. **24:3** (1990), 14–25 (Russian). Translation in Funct. Anal. Appl. **24:3** (1990), 176186.

[7] W. Kirsch, S.A. Molchanov, L.A. Pastur, *One-dimensional Schrdinger operators with high potential barriers*. Operator calculus and spectral theory (Lambrecht, 1991), 163 – 170, Oper. Theory Adv. Appl., 57, Birkhuser, Basel, 1992.

[8] A. Kiselev, Y. Last, B. Simon, *Modified Prüfer and EFGP transforms and the spectral analysis of one-dimensional Schrödinger operators*, Comm. Math. Phys. 194 (1998), 1-45.

[9] H. Krüger, *A family of Schrödinger Operators whose spectrum is an interval*, Comm. Math. Phys. 290:3, 935-939 (2009).

[10] H. Krüger, *Probabilistic averages of Jacobi operators.* Comm. Math. Phys. **295:3**, 853–875 (2010).

[11] H. Krüger, *Multiscale Analysis for Ergodic Schrödinger operators and positivity of Lyapunov exponents*, J. d'Analyse Math. (to appear).

[12] H. Krüger, *Semiclassical analysis of the largest gap of quasi-periodic Schrödinger operators.* Math. Model. Nat. Phenom. **5:4**, 256-268 (2010).

[13] M. Lukic, *Orthogonal polynomials with recursion coefficients of generalized bounded variation.*

[14] J. von Neumann, E. Wigner, *Über merkwürdige diskrete Eigenwerte.* Phys. Z. **30** (1929), 467–470.

[15] C. Remling, *The absolutely continuous spectrum of one-dimensional Schrödinger operators with decaying potentials.* Comm. Math. Phys. **193** (1998), 151 – 170.

[16] B. Simon, *Some Jacobi matrices with decaying potential and dense point spectrum*, Comm. Math. Phys. **87** (1982), 253–258.

[17] B. Simon, *Trace Ideals and Their Applications.* Second edition, Mathematical Surveys and Monographs, **120**, American Mathematical Society, 2005

[18] G. Stolz, *Spectral theory for slowly oscillating potentials. I. Jacobi matrices.* Manuscripta Math. **84**, no. 3-4, 245–260 (1994).

[19] G. Stolz, *Localization for Schrödinger operators with effective barriers*, J. Funct. Anal. **146:2** (1997), 416–429.

[20] G. Teschl, *Jacobi Operators and Completely Integrable Nonlinear Lattices.* Math. Surv. and Mon. **72**, Amer. Math. Soc., Rhode Island, 2000.

MATHEMATICS 253-37, CALTECH, PASADENA, CA 91125
E-mail address: `helge@caltech.edu`
URL: `http://www.its.caltech.edu/~helge/`